

# −1-PHENOMENA FOR THE PLURI $\chi_y$ -GENUS AND ELLIPTIC GENUS

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**ABSTRACT.** Several independent articles have observed that the Hirzebruch  $\chi_y$ -genus has an important feature, which the author calls  $-1$ -phenomenon and tells us that the coefficients of the Taylor expansion of the  $\chi_y$ -genus at  $y = -1$  have explicit expressions. Hirzebruch's original  $\chi_y$ -genus can be extended towards two directions: the pluri-case and the case of elliptic genus. This paper contains two parts in which we investigate the  $-1$ -phenomena in these two generalized cases respectively and show that in each case there exists a  $-1$ -phenomenon in a suitable sense. Our main results in the first part have an application, which states that all characteristic numbers (Chern numbers and Pontrjagin numbers) on manifolds can be expressed, in a very explicit way, in terms of some rationally linear combination of indices of some elliptic operators. This gives an analytic interpretation of characteristic numbers and affirmatively answers a question posed by the author several years ago. The second part contains our attempt to generalize this  $-1$ -phenomenon to elliptic genus, a modern version of the  $\chi_y$ -genus. We first extend the elliptic genus of an almost-complex manifold to a twisted version where an extra complex vector bundle is involved, and show that it is a weak Jacobi form under some assumptions. A suitable manipulation on the theory of Jacobi form will produce new modular forms from this weak Jacobi form and thus much arithmetic information related to the underlying manifold can be obtained, in which the  $-1$ -phenomenon of the original  $\chi_y$ -genus is hidden.

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## 1. INTRODUCTION

**1.1. The Hirzebruch  $\chi_y$ -genus and its  $-1$ -phenomenon.** In his highly influential book [15], Hirzebruch defined a polynomial with integral coefficients  $\chi_y(M)$  for projective manifolds  $M$ , which encodes the information of indices of Dolbeault complexes and is now called the *Hirzebruch  $\chi_y$ -genus*. After the discovery of the general index theorem due to Atiyah and Singer, we know that  $\chi_y(\cdot)$  can be defined on compact almost-complex manifolds and computed in terms of Chern numbers as follows.

Suppose  $(M^{2d}, J)$  is a compact connected almost-complex manifold with an almost-complex structure  $J$ . The choice of an almost Hermitian metric on  $M$  enables us to define the Hodge star operator  $*$  and the formal adjoint  $\bar{\partial}^* = - * \bar{\partial} *$  of the  $\bar{\partial}$ -operator. For each pair  $0 \leq p, q \leq d$ , we denote by

$$\Omega^{p,q}(M) := \Gamma(\Lambda^p T^* M \otimes \Lambda^q \overline{T^* M})$$

the complex vector space which consists of smooth complex-valued  $(p, q)$ -forms. Here  $T^* M$  is the dual of holomorphic tangent bundle  $TM$  in the sense of  $J$ . Then for each  $0 \leq p \leq d$ , we have the following Dolbeault-type elliptic differential operator

$$\bigoplus_{q \text{ even}} \Omega^{p,q}(M) \xrightarrow{(\bar{\partial} + \bar{\partial}^*)|_p} \bigoplus_{q \text{ odd}} \Omega^{p,q}(M),$$

whose index is denoted by  $\chi^p(M)$  in the notation of Hirzebruch [15]. Then the Hirzebruch  $\chi_y$ -genus of  $M$  is nothing but the generating function of these indices  $\chi^p(M)$  ( $0 \leq p \leq d$ ):

$$\chi_y(M) := \sum_{p=0}^d \chi^p(M) \cdot y^p.$$

If we denote by  $x_1, \dots, x_d$  the formal Chern roots of  $TM$ , i.e., the  $i$ -th elementary symmetric polynomial of  $x_1, \dots, x_d$  represents  $c_i$ , the  $i$ -th Chern class of  $TM$ , then the general form of the Hirzebruch-Riemann-Roch theorem (first proved by Hirzebruch for projective manifolds [15], and in the general case by Atiyah and Singer [2]) tells us that

$$(1.1) \quad \chi_y(M) = \int_M \prod_{i=1}^d \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}}.$$

Among other things, the Hirzebruch  $\chi_y$ -genus has an important feature, which the author calls “ $-1$ -phenomenon” and has been noticed, implicitly or explicitly, in several independent articles ([28], [20], [29]). This  $-1$ -phenomenon says that, at  $y = -1$ , the coefficients of the Taylor expansion of  $\chi_y(M)$  have explicit expressions. To be more precise, if we write

$$(1.2) \quad \chi_y(M) =: \sum_{i=0}^d a_i(M) \cdot (y + 1)^i,$$

then these  $a_i(M)$  can be given explicit expressions in terms of Chern numbers of  $(M^{2d}, J)$  as follows.

$$(1.3) \quad \begin{aligned} a_0(M) &= c_d, & a_1(M) &= -\frac{1}{2}dc_d, & a_2(M) &= \frac{1}{12}\left[\frac{d(3d-5)}{2}c_d + c_1c_{d-1}\right], \\ a_3(M) &= -\frac{1}{24}\left[\frac{d(d-2)(d-3)}{2}c_d + (d-2)c_1c_{d-1}\right], & \dots \end{aligned}$$

By definition, these  $a_i(M)$  are integers. Thus immediate consequences of their expressions are divisibility properties of Chern numbers. The derivation of these expressions is direct. That is, to expand the right-hand side of (1.1) at  $y = -1$  and express the coefficients in terms of elementary symmetric polynomials of  $x_1, \dots, x_d$ . The calculations of  $a_0$  and  $a_1$  are quite easy. The calculation of  $a_2$  appears implicitly in [28, p. 18] and explicitly in [20, p. 141-143]. Narasimhan and Ramanan used  $a_2$  to give a topological restriction on some moduli spaces of stable vector bundles on smooth projective varieties. Libgober and Wood used  $a_2$  to prove the uniqueness of the complex structure on Kähler manifolds of certain homotopy types. Inspired by [28], S. Salamon applied  $a_2$  [29, Corollary 3.4] to obtain a restriction on the Betti numbers of hyper-Kähler manifolds [29, Theorem 4.1]. The expressions of  $a_3$  and  $a_4$  are also included in [29, p. 145]. Hirzebruch used  $a_1$ ,  $a_2$  and  $a_3$  to obtain a divisibility result on the Euler characteristic of those almost-complex manifolds whose  $c_1c_{d-1} = 0$  ([16]). In particular, those almost-complex manifolds with  $c_1 = 0$  satisfy this property.

**1.2. Pluri- $\chi_y$ -genus.** Some acquaintance with index theory will lead to the observation that  $\chi_y(M)$  is the index of the following Todd operator whose index is the Todd genus

$$(1.4) \quad \Omega^{0,\text{even}}(M) \xrightarrow{(\bar{\partial} + \bar{\partial}^*)|_0} \Omega^{0,\text{odd}}(M)$$

twisted by  $\Omega_y(M)$ , where

$$\Omega_y(M) := \sum_{p=0}^d \Lambda^p(T^*M) \cdot y^p \in K(M)[y]$$

and  $\Lambda^p(\cdot)$  (resp.  $K(\cdot)$ ) denotes the  $p$ -th exterior power (resp. K-group). Therefore  $\chi_y(M)$  can be rewritten as follows:

$$\chi_y(M) = \text{Ind}((\bar{\partial} + \bar{\partial}^*)|_0 \otimes \Omega_y(M)) =: \chi(M, \Omega_y(M)).$$

Here for simplicity we denote by the standard notation  $\chi(M, (\cdot))$  the index of the Todd operator (1.4) twisted by an element  $(\cdot) \in K(M)$ .

We can also consider, for arbitrarily fixed positive integer  $g$ , the *pluri  $\chi_y$ -genus*  $\chi_{\underline{y}}(M)$  by using sufficiently many forms of the type

$$\begin{aligned} \Omega_{\underline{y}}(M) &:= \sum_{0 \leq p_1, \dots, p_g \leq d} \Lambda^{p_1}(T^*M) \otimes \dots \otimes \Lambda^{p_g}(T^*M) \cdot y_1^{p_1} \dots y_g^{p_g} \\ &= \Omega_{y_1}(M) \otimes \dots \otimes \Omega_{y_g}(M) \in K(M)[y_1, \dots, y_g] \end{aligned}$$

to twist  $(\bar{\partial} + \bar{\partial}^*)|_0$ , i.e.,

$$\chi_{\underline{y}}(M) := \text{Ind}((\bar{\partial} + \bar{\partial}^*)|_0 \otimes \Omega_{\underline{y}}(M)) = \chi(M, \Omega_{\underline{y}}(M)),$$

which specializes to the Hirzebruch's original  $\chi_y$ -genus when  $g = 1$ .

Inspired by the above-mentioned  $-1$ -phenomenon of the  $\chi_y$ -genus, we may ask what the coefficients look like if we expand  $\chi_{\underline{y}}(M)$  at  $y_1 = \dots = y_g = -1$ . Our first main observation in

this article is that the coefficients of  $(y+1)^{p_1} \cdots (y+1)^{p_g}$  in  $\chi_y(M)$  can be divided into three parts, which is our main result in Section 3 (Theorem 2.2). Moreover, we can do a similar manipulation for signature operator on closed smooth oriented manifolds and their coefficients also have a similar feature (Theorem 2.3). A direct corollary of these two theorems is that any Chern number of  $(M^{2d}, J)$  (resp. any Pontrjagin number of a closed smooth oriented manifold) can be written as a rationally linear combination of indices of some elliptic operators explicitly, which provides an analytic interpretation of characteristic numbers and answers a question of the author proposed in [19, Question 1.1] affirmatively.

**1.3. Elliptic genus.** Elliptic genera of oriented differentiable manifolds and almost-complex manifolds were first constructed by Ochanine, Landweber-Stong and Hirzebruch in a topological way and Witten gave it a geometric interpretation, which can be viewed as loop spaces' analogues to the Hirzebruch  $L$ -genus and  $\chi_y$ -genus (cf. [18] and the references therein). The most remarkable property of elliptic genera is their rigidity for spin manifolds and almost-complex Calabi-Yau manifolds (in the very weak sense that  $c_1$  vanishes up to torsion, i.e.,  $c_1 = 0 \in H^2(M, \mathbb{R})$ ), which was conjectured by Witten and generalizes the famous rigidity property of the original  $L$ -genus,  $\hat{A}$ -genus ([1]) and  $\chi_y$ -genus ([27]). The first rigorous proof was presented by Bott and Taubes ([30], [4]). A quite simple, unified and enlightening proof was discovered by Liu ([22]), in which modular invariance of the four classical Jacobi-theta functions and their various transformation laws play key roles. Later on, this modular invariance property, its variously remarkable extensions and relation with vertex operator algebra were established by Liu and his coauthors from various aspects ([21], [23], [24], [25], [26], [8], [6], [7], [12], [13], [14] etc.).

What we are concerned with in this paper is the elliptic genus of almost-complex manifolds. The elliptic genus of a compact, almost-complex manifold  $(M^{2d}, J)$ , which we denote by  $\text{Ell}(M, \tau, z)$ , is defined as a function of two variables  $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ , where  $\mathbb{H}$  is the upper half plane. To be more precise,  $\text{Ell}(M, \tau, z)$  is defined to be the index of the Todd operator (1.4) twisted by

$$y^{-\frac{d}{2}} \otimes_{n \geq 1} (\Lambda_{-yq^{n-1}} T^* \otimes \Lambda_{-y^{-1}q^n} T \otimes S_{q^n} T^* \otimes S_{q^n} T) =: E_{q,y},$$

i.e.,  $\text{Ell}(M, \tau, z) := \chi(M, E_{q,y})$ , where  $q = e^{2\pi\sqrt{-1}\tau}$ ,  $y = e^{2\pi\sqrt{-1}z}$  and  $T$  (resp.  $T^*$ ) is the holomorphic (resp. dual of holomorphic) tangent bundle of  $M$  in the sense of  $J$ . Here

$$\Lambda_t(W) := \bigoplus_{i \geq 0} \Lambda^i(W) \quad \text{and} \quad S_t(W) := \bigoplus_{i \geq 0} S^i(W)$$

for any complex vector bundle  $W$  denote the generating series of the exterior and symmetric powers of  $W$  respectively.

According to the Atiyah-Singer index theorem we have

$$\begin{aligned} \text{Ell}(M, \tau, z) &= \int_M \text{td}(M) \cdot \text{ch}(E_{q,y}) \\ &= y^{-\frac{d}{2}} \chi_{-y}(M) + q \cdot [y^{-\frac{d}{2}} \chi_{-y}(M, T^*(1-y) + T(1-y^{-1}))] + q^2 \cdot (\cdots), \end{aligned}$$

where

$$\text{td}(M) := \prod_{i=1}^d \frac{x_i}{1 - e^{-x_i}}$$

is the Todd class of  $M$  and  $\text{ch}(\cdot)$  is the Chern character.

Thus elliptic genus  $\text{Ell}(M, \tau, z)$  can be viewed as a generalization of the Hirzebruch  $\chi_y$ -genus in the sense that the  $q^0$ -term of the Fourier expansion of  $\text{Ell}(M, \tau, z)$  is essentially  $\chi_y(M)$ . If  $(M^{2d}, J)$  is Calabi-Yau, the coefficients of  $q$ -expansion of  $\text{Ell}(M, \tau, z)$  are rigid for arbitrary  $y$  ([22, Theorem B]). Moreover, in this case,  $\text{Ell}(M, \tau, z)$  itself is a weak Jacobi form of weight 0 and index  $\frac{d}{2}$  ([3, Theorem 2.2], [10, Proposition 1.2]).

As we have mentioned above, elliptic genus  $\text{Ell}(M, \tau, z)$  can be viewed as a generalization of  $\chi_y(M)$  and also has a rigidity property when  $M$  is Calabi-Yau. So we may ask, in the case of Calabi-Yau, whether  $\text{Ell}(M, \tau, z)$  has some kind of arithmetic phenomenon which extends the original  $-1$ -phenomenon of  $\chi_y(M)$ . Note that, strictly speaking,  $\text{Ell}(M, \tau, z)$  is a generalization of  $\chi_{-y}(M)$  rather than  $\chi_y(M)$  as the  $q^0$ -term of  $\text{Ell}(M, \tau, z)$  is  $y^{-\frac{d}{2}}\chi_{-y}(M)$ . So if there exists some kind of phenomenon which extends the original  $-1$ -phenomenon of  $\chi_y(M)$ , the parameter  $y = e^{2\pi\sqrt{-1}z}$  should correspond to 1 rather than  $-1$ . Thus the variable  $z$  should correspond to 0. Indeed, there does exist such a kind of generalization, which depends on some arithmetic properties of Jacobi form and has been implicitly used by Gritsenko in [10]. Our aim in Section 3 is two-fold. On the one hand, given a compact almost-complex manifold  $(M^{2d}, J)$  and a rank  $l$  complex vector bundle  $W$  over it, we construct a generalized elliptic genus  $\text{Ell}(M, W, \tau, z)$ , which is defined to be the index of the Todd operator (1.4) twisted by

$$\left[\prod_{i=1}^{\infty} (1 - q^i)\right]^{2(d-l)} \cdot y^{-\frac{l}{2}} \otimes_{n \geq 1} (\Lambda_{-yq^{n-1}} W^* \otimes \Lambda_{-y^{-1}q^n} W \otimes S_{q^n} T^* \otimes S_{q^n} T),$$

and show that it is a weak Jacobi form of weight  $d - l$  and index  $\frac{l}{2}$  if the first Pontrjagin classes  $p_1(M) = p_1(W)$  and the first Chern class  $c_1(W) = 0$  in  $H^*(M, \mathbb{R})$ . On the other hand, we highlight a well-known manipulation in Jacobi form to obtain modular forms from  $\text{Ell}(M, W, \tau, z)$ , whose arithmetic information will in turn give geometric results on  $M$  and  $W$ . Some examples are given to illustrate this observation.

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## 2. -1-PHENOMENON OF THE PLURI- $\chi_y$ -GENUS

**2.1. Statements of the main results related to the pluri- $\chi_y$ -genus.** Let  $(M^{2n}, J)$  (resp.  $X^{2n}$ ) be a compact almost-complex manifold of complex dimension  $n$  (resp. smooth, closed oriented manifold of real dimension  $2n$ ). As before we use  $(\bar{\partial} + \bar{\partial}^*)|_0$  to denote the Todd operator on  $(M^{2n}, J)$  whose index is the Todd genus of  $M$ . We denote by  $D$  the signature operator on  $X$ , whose index is the signature of  $X^{2n}$  ([2, §6]). By definition  $\text{Ind}(D)$  is zero unless  $n$  is even.

Let  $W$  be a complex vector bundle over  $M$  (resp.  $X$ ). By means of a connection on  $W$ , the elliptic operator  $(\bar{\partial} + \bar{\partial}^*)|_0$  and  $D$  can be extended to a new elliptic operator  $((\bar{\partial} + \bar{\partial}^*)|_0) \otimes W$

and  $D \otimes W$ , whose indices via the Atiyah-Singer index theorem are

$$\begin{aligned}\chi(M, W) &= \text{Ind}((\bar{\partial} + \bar{\partial}^*)|_0 \otimes W) = \int_M [\text{td}(M) \cdot \text{ch}(W)] \\ &= \int_M \left[ \prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}} \cdot \text{ch}(W) \right]\end{aligned}$$

and

$$\text{Ind}((D \otimes W)) = \int_X \left[ \left( \prod_{i=1}^n \frac{x_i}{\tanh \frac{x_i}{2}} \right) \cdot \text{ch}(W) \right]$$

respectively. Here we use the  $i$ -th elementary symmetry polynomial of  $x_1, \dots, x_n$  (resp.  $x_1^2, \dots, x_n^2$ ) to denote the  $i$ -th Chern class (resp. Pontrjagin class) of  $(M^{2n}, J)$  (resp.  $X^{2n}$ ).

**Definition 2.1.** For arbitrarily fixed positive integer  $g$ , we define

$$\begin{aligned}\Omega_{\underline{y}}(M) &:= \sum_{0 \leq p_1, \dots, p_g \leq n} \Lambda^{p_1}(T^*M) \otimes \dots \otimes \Lambda^{p_g}(T^*M) \cdot y_1^{p_1} \dots y_g^{p_g} \\ &= \Omega_{y_1}(M) \otimes \dots \otimes \Omega_{y_g}(M) \in K(M)[y_1, \dots, y_g],\end{aligned}$$

$$\begin{aligned}\Omega_{\underline{y}}^{\mathbb{R}}(X) &:= \sum_{0 \leq p_1, \dots, p_g \leq 2n} \Lambda^{p_1}(T_{\mathbb{C}}^*X) \otimes \dots \otimes \Lambda^{p_g}(T_{\mathbb{C}}^*X) \cdot y_1^{p_1} \dots y_g^{p_g} \\ &= \Omega_{y_1}^{\mathbb{R}}(X) \otimes \dots \otimes \Omega_{y_g}^{\mathbb{R}}(X) \in (KO(X) \otimes \mathbb{C})[y_1, \dots, y_g],\end{aligned}$$

where

$$\Omega_{\underline{y}}^{\mathbb{R}}(X) := \sum_{p=0}^{2n} \Lambda^p(T_{\mathbb{C}}^*X) \cdot y^p$$

and  $T_{\mathbb{C}}^*X$  is the dual of the complexified tangent bundle of  $X$ , and

$$\begin{aligned}\chi_{\underline{y}}(M) &:= \sum_{0 \leq p_1, \dots, p_g \leq n} \text{Ind}[(\bar{\partial} + \bar{\partial}^*)|_0 \otimes (\Lambda^{p_1}(T^*M) \otimes \dots \otimes \Lambda^{p_g}(T^*M))] \cdot y_1^{p_1} \dots y_g^{p_g} \\ &= \int_M \left[ \prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}} \cdot \text{ch}(\Omega_{\underline{y}}(M)) \right]\end{aligned}$$

$$\begin{aligned}D_{\underline{y}}(X) &:= \sum_{0 \leq p_1, \dots, p_g \leq 2n} \text{Ind}[D \otimes (\Lambda^{p_1}(T_{\mathbb{C}}^*X) \otimes \dots \otimes \Lambda^{p_g}(T_{\mathbb{C}}^*X))] \cdot y_1^{p_1} \dots y_g^{p_g} \\ &= \int_X \left[ \left( \prod_{i=1}^n \frac{x_i}{\tanh \frac{x_i}{2}} \right) \cdot \text{ch}(\Omega_{\underline{y}}^{\mathbb{R}}(X)) \right].\end{aligned}$$

Our main result in this section is

**Theorem 2.2.** The coefficient of  $(1 + y_1)^{n-q_1} \dots (1 + y_g)^{n-q_g}$  in  $\chi_{\underline{y}}(M)$  is equal to

$$\begin{cases} 0, & \text{if } \sum_{i=1}^g q_i > n, \\ \int_M \prod_{i=1}^g c_{q_i}(M), & \text{if } \sum_{i=1}^g q_i = n, \\ \text{a rationally linear combination of Chern numbers of } M, & \text{if } \sum_{i=1}^g q_i < n. \end{cases}$$

We have a similar result for smooth manifolds.

**Theorem 2.3.** If  $n$  is even, the coefficient of

$$(1 + y_1)^{2(n-q_1)} \cdots (1 + y_g)^{2(n-q_g)}$$

in  $D_{\underline{y}}(X)$  is equal to

$$\begin{cases} 0, & \text{if } \sum_{i=1}^g q_i > \frac{n}{2}, \\ (-1)^{\frac{n}{2}} \cdot 2^n \cdot \int_X \prod_{i=1}^g p_{q_i}(X), & \text{if } \sum_{i=1}^g q_i = \frac{n}{2}, \\ \text{a rationally linear combination of Pontrjagin numbers of } X, & \text{if } \sum_{i=1}^g q_i < \frac{n}{2}, \end{cases}$$

where  $p_i(X)$  is the  $i$ -th Pontrjagin class of  $X$ .

Clearly a direct corollary of this theorem is the following result, which gives an affirmative answer to a question proposed by the author in [19, Question 1.1].

**Corollary 2.4.** Any Chern number (resp. Pontrjagin number) on a compact almost-complex manifold (resp. compact smooth manifold) can be expressed, in a very explicit way, in terms of the indices of some elliptic differential operators over this manifold.

**2.2. Proofs of Theorems 2.2 and 2.3.** By the abuse of notation we use  $c_q(\cdots)$  to denote both the  $q$ -th Chern class of an almost-complex manifold and the  $q$ -th elementary symmetric polynomial of the variables in the bracket.

The proofs of Theorems 2.2 and 2.3 depend on the following lemma.

**Lemma 2.5.** If we assign each  $x_i$  ( $1 \leq i \leq n$ ) the same degree, then we have

- (1) The coefficient of  $(1 + y)^{n-q}$  ( $0 \leq q \leq n$ ) in  $\prod_{i=1}^n (1 + ye^{-x_i})$  is
$$c_q(x_1, \dots, x_n) + \text{higher degree terms.}$$
- (2) The coefficient of  $(1 + y)^{2(n-q)}$  ( $0 \leq q \leq n$ ) in  $\prod_{i=1}^n (1 + ye^{-x_i})(1 + ye^{x_i})$  is
$$(-1)^q c_q(x_1^2, \dots, x_n^2) + \text{higher degree terms.}$$

*Proof.*

$$\prod_{i=1}^n (1 + ye^{-x_i}) = \prod_{i=1}^n [(1 - e^{-x_i}) + (1 + y)e^{-x_i}] = e^{-c_1} \prod_{i=1}^n [(e^{x_i} - 1) + (1 + y)].$$

Thus the coefficient of  $(1 + y)^{n-q}$  in  $\prod_{i=1}^n (1 + ye^{-x_i})$  is

$$e^{-c_1} \cdot c_q(e^{x_1} - 1, \dots, e^{x_n} - 1) = c_q(x_1, \dots, x_n) + \text{higher degree terms.}$$

Similarly,

$$\prod_{i=1}^n (1 + ye^{-x_i})(1 + ye^{x_i}) = \prod_{i=1}^n [(e^{x_i} - 1) + (1 + y)][(e^{-x_i} - 1) + (1 + y)]$$

and the coefficient of  $(1 + y)^{2n-q}$  is

$$\begin{aligned} & c_q(e^{x_1} - 1, \dots, e^{x_n} - 1, e^{-x_1} - 1, \dots, e^{-x_n} - 1) \\ &= c_q(x_1, \dots, x_n, -x_1, \dots, -x_n) + \text{higher degree terms} \end{aligned}$$

Note that

$$c_q(x_1, \dots, x_n, -x_1, \dots, -x_n) = \begin{cases} 0, & \text{if } q \text{ is odd,} \\ (-1)^{\frac{q}{2}} c_{\frac{q}{2}}(x_1^2, \dots, x_n^2), & \text{if } q \text{ is even.} \end{cases}$$

This gives the desired property.  $\square$

Now we can prove Theorems 2.2 and 2.3.

*Proof.* If we use  $x_1, \dots, x_n$  (resp.  $x_1, \dots, x_n, -x_1, \dots, -x_n$ ) to denote the formal Chern roots of  $TM$  (resp.  $T_{\mathbb{C}}X$ ), then we have (cf. [17, p. 11])

$$\text{ch}(\Omega_{\underline{y}}(M)) = \prod_{j=1}^g \left[ \prod_{i=1}^n (1 + y_j e^{-x_i}) \right]$$

and

$$\text{ch}(\Omega_{\underline{y}}^{\mathbb{R}}(X)) = \prod_{j=1}^g \left[ \prod_{i=1}^n (1 + y_j e^{-x_i})(1 + y_j e^{x_i}) \right].$$

Thus

$$\begin{aligned} \chi_{\underline{y}}(M) &= \int_M \left[ \left( \prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}} \right) \cdot \text{ch}(\Omega_{\underline{y}}(M)) \right] \\ &= \int_M \left\{ \left( \prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}} \right) \cdot \prod_{j=1}^g \left[ \prod_{i=1}^n (1 + y_j e^{-x_i}) \right] \right\} \end{aligned}$$

and

$$\begin{aligned} \text{ind}(D_{\underline{y}}^{\mathbb{R}}(X)) &= \int_X \left[ \left( \prod_{i=1}^n \frac{x_i}{\tanh \frac{x_i}{2}} \right) \cdot \text{ch}(\Omega_{\underline{y}}^{\mathbb{R}}(X)) \right] \\ &= \int_X \left\{ \left( \prod_{i=1}^n \frac{x_i}{\tanh \frac{x_i}{2}} \right) \cdot \prod_{j=1}^g \left[ \prod_{i=1}^n (1 + y_j e^{-x_i})(1 + y_j e^{x_i}) \right] \right\}. \end{aligned}$$

$\square$

Note that the constant terms of

$$\frac{x_i}{1 - e^{-x_i}} = 1 + \dots$$

and

$$\frac{x_i}{\tanh \frac{x_i}{2}} = \frac{x_i(1 + e^{-x_i})}{1 - e^{-x_i}} = 2 + \dots$$

are 1 and 2 respectively. So by Lemma 2.5, when considering the Taylor expansion of  $\text{ind}(D_{\underline{y}}(M))$  (resp.  $\text{ind}(D_{\underline{y}}^{\mathbb{R}}(X))$ ) at  $y_1 = \dots = y_g = -1$ , the coefficients before the terms  $(1 + y_1)^{n-q_1} \dots (1 + y_g)^{n-q_g}$  (resp.  $(1 + y_1)^{2(n-q_1)} \dots (1 + y_g)^{2(n-q_g)}$ ) are

$$\begin{aligned} &\int_M \left\{ (1 + \text{higher degree terms}) \cdot \prod_{j=1}^g [c_{q_j}(x_1, \dots, x_n) + \text{higher degree terms}] \right\} \\ &= \int_M \prod_{i=1}^g c_{q_i}(M) + \int_M (\text{higher degree terms}), \end{aligned}$$



$$\begin{aligned}
& \left( \text{resp. } \int_X \{ (2^n + \text{higher degree terms}) \cdot \prod_{j=1}^g [(-1)^{q_j} c_{q_j}(x_1^2, \dots, x_n^2) + \text{higher degree terms}] \} \right. \\
& \quad \left. = 2^n \cdot (-1)^{\sum_{j=1}^g q_j} \int_X \prod_{j=1}^g p_{q_j}(X) + \int_X (\text{higher degree terms}), \right)
\end{aligned}$$

which give the desired proofs of Theorems 2.2 and 2.3.

### 3. GENERALIZED ELLIPTIC GENUS AND ITS -1-PHENOMENON

**3.1. Generalized elliptic genus of almost-complex manifolds.** In this subsection we extend the original definition of elliptic genus of almost-complex manifolds by considering an extra complex vector bundle and show that it is a weak Jacobi form. As before, let  $(M^{2d}, J)$  be a compact almost-complex manifold and  $W$  be a rank  $l$  complex vector bundle over it.

**Definition 3.1.** The generalized elliptic genus of  $(M^{2d}, J)$  with respect to  $W$ , which we denote by  $\text{Ell}(M, W, \tau, z)$ , is defined to be the index of the Todd operator

$$\Omega^{0, \text{even}}(M) \xrightarrow{(\bar{\partial} + \bar{\partial}^*)|_0} \Omega^{0, \text{odd}}(M)$$

twisted by

$$c^{2(d-l)} \cdot y^{-\frac{l}{2}} \otimes_{n \geq 1} (\Lambda_{-yq^{n-1}} W^* \otimes \Lambda_{-y^{-1}q^n} W \otimes S_{q^n} T^* \otimes S_{q^n} T) =: E(W, q, y),$$

where

$$q = e^{2\pi\sqrt{-1}\tau}, \quad y = e^{2\pi\sqrt{-1}z},$$

and for simplicity  $c := \prod_{i=1}^{\infty} (1 - q^i)$ .

If  $W = T$ , this definition degenerates to the original elliptic genus.

Our first observation in this section is the following result, which extends [3, Theorem 2.2] and [10, Proposition 1.2] in which case  $W = T$ .

**Theorem 3.2.** The generalized elliptic genus  $\text{Ell}(M, W, \tau, z)$  is a weak Jacobi form of weight  $d - l$  and index  $\frac{l}{2}$  provided that the first Pontrjagin classes  $p_1(M) = p_1(W)$  and the first Chern class  $c_1(W) = 0$  in  $H^*(M, \mathbb{R})$ .

**Remark 3.3.**

- (1) A two-variable function  $\varphi(\tau, z)$  for  $(\tau, z) \in \mathbb{H} \times \mathbb{C}$  is called a *weak Jacobi form of weight  $k$  and index  $m$*  for  $k \in \mathbb{Z}$  and  $m \in \mathbb{Z}/2$  if it is a holomorphic function with respect to the two variables  $\tau$  and  $z$ , has no negative powers of  $q$  in its Fourier expansion in terms of  $q^i y^j$  and satisfies some transformation laws involving  $k$  and  $m$ , where the precise definition can be found in [9, p. 104, p. 9]. In [9] the only integral indices are considered. However, with some minor modifications of inserting a character, this notion can be easily extended to the case where the index is allowed to be a half-integer. (cf. [10, p. 102]).
- (2) Motivated by his ingenious proof of the rigidity theorem, Liu constructed a two-variable function for  $(M, J)$  and  $W$  and showed that it is a weak Jacobi form under some assumptions and the original Witten theorem exactly corresponds to the case where the index is equal to zero ([23, Theorem 3, Corollary 3.1]). This construction

later was generalized to the family case by Liu-Ma ([24, Theorem 3.1]). So our theorem has a similar flavor to those of Liu and Ma.

- (3) In [11, Theorem 1.2], Gritsenko further extended the original elliptic genus to another case where an extra complex bundle is involved. But his construction is different from ours as it is still of weight zero.

The Atiyah-Singer index theorem tells us that

$$\text{Ell}(M, W, \tau, z) = \int_M \text{td}(M) \cdot \text{ch}(E(W, q, y)).$$

In particular, if  $J$  is integrable,  $\text{Ell}(M, W, \tau, z)$  is the holomorphic Euler characteristic of the (virtual) bundle  $E(W, q, y)$ .

Let us recall one of the famous Jacobi-theta series ([5, Chapter 5])

$$\begin{aligned} \theta(\tau, z) &:= \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{(n+\frac{1}{2})^2}{2}} y^{n+\frac{1}{2}} \\ &= 2cq^{\frac{1}{8}} \sin(\pi z) \prod_{n=1}^{\infty} (1 - q^n y)(1 - q^n y^{-1}) \\ &= 2cq^{\frac{1}{8}} \sinh(\pi \sqrt{-1} z) \prod_{n=1}^{\infty} (1 - q^n y)(1 - q^n y^{-1}) \\ &= 2cq^{\frac{1}{8}} \sinh(\pi \sqrt{-1} z) \prod_{n=1}^{\infty} (1 - q^n e^{2\pi \sqrt{-1} z})(1 - q^n e^{-2\pi \sqrt{-1} z}). \end{aligned}$$

The following lemma says that  $\text{Ell}(M, W, \tau, z)$  can be expressed in terms of  $\theta(\tau, z)$ .

**Lemma 3.4.** If we denote by  $2\pi\sqrt{-1}x_i$  ( $1 \leq i \leq d$ ) and  $2\pi\sqrt{-1}w_i$  ( $1 \leq i \leq l$ ) respectively the Chern roots of  $TM$  and  $W$ , then we have

$$\begin{aligned} &\text{Ell}(M, W, \tau, z) \\ &= \int_M \left[ \exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \cdot (\eta(\tau))^{3(d-l)} \cdot \prod_{i=1}^d \frac{2\pi\sqrt{-1}x_i}{\theta(\tau, x_i)} \cdot \prod_{j=1}^l \theta(\tau, w_j - z) \right], \end{aligned}$$

where

$$\eta(\tau) := q^{\frac{1}{24}} \cdot c = q^{\frac{1}{24}} \prod_{i=1}^{\infty} (1 - q^i)$$

is the famous Dedekind eta function. In particular,  $\text{Ell}(M, W, \tau, z)$  is a holomorphic function with respect to the two variables  $\tau$  and  $z$  and has no negative powers of  $q$  in its Fourier expansion.

*Proof.*

$$\begin{aligned}
& \text{ch}(\mathbf{E}(W, q, y)) \\
&= c^{2(d-l)} y^{-\frac{l}{2}} \prod_{j=1}^l (1 - ye^{-2\pi\sqrt{-1}w_j}) \prod_{n=1}^{\infty} \frac{\prod_{j=1}^l (1 - yq^n e^{-2\pi\sqrt{-1}w_j})(1 - y^{-1}q^n e^{2\pi\sqrt{-1}w_j})}{\prod_{i=1}^d (1 - q^n e^{-2\pi\sqrt{-1}x_i})(1 - q^n e^{2\pi\sqrt{-1}x_i})} \\
&= c^{2(d-l)} y^{-\frac{l}{2}} \prod_{j=1}^l (1 - ye^{-2\pi\sqrt{-1}w_j}) \prod_{j=1}^l \frac{\theta(\tau, w_j - z)}{2cq^{\frac{1}{8}} \sinh(\pi\sqrt{-1}(w_j - z))} \prod_{i=1}^d \frac{2cq^{\frac{1}{8}} \sinh(\pi\sqrt{-1}x_i)}{\theta(\tau, x_i)} \\
&= \exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \cdot (\eta(\tau))^{3(d-l)} \cdot \prod_{i=1}^d \frac{1 - e^{-2\pi\sqrt{-1}x_i}}{\theta(\tau, x_i)} \cdot \prod_{j=1}^l \theta(\tau, w_j - z)
\end{aligned}$$

The last equality is due to the facts that

$$c_1(M) = \sum_{i=1}^d 2\pi\sqrt{-1}x_i \quad \text{and} \quad c_1(W) = \sum_{j=1}^l 2\pi\sqrt{-1}w_j.$$

Therefore,

$$\begin{aligned}
& \text{Ell}(M, W, \tau, z) \\
&= \int_M \text{td}(M) \cdot \text{ch}(\mathbf{E}(W, q, y)) \\
&= \int_M \prod_{i=1}^d \frac{2\pi\sqrt{-1}x_i}{1 - e^{-2\pi\sqrt{-1}x_i}} \cdot \text{ch}(\mathbf{E}(W, q, y)) \\
&= \int_M \left[ \exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \cdot (\eta(\tau))^{3(d-l)} \cdot \prod_{i=1}^d \frac{2\pi\sqrt{-1}x_i}{\theta(\tau, x_i)} \cdot \prod_{j=1}^l \theta(\tau, w_j - z) \right]
\end{aligned}$$

The holomorphicity of  $\text{Ell}(M, W, \tau, z)$  is now clear from this expression as the Jacobi-theta function  $\theta(\tau, z)$  only has zeroes of order 1 along  $z = m_1 + m_2\tau$  ( $m_1, m_2 \in \mathbb{Z}$ ) ([5, p. 59]). Also it is obvious from this expression that  $\text{Ell}(M, W, \tau, z)$  has no negative powers of  $q$  in its Fourier expansion.  $\square$

*Proof of Theorem 3.2.*

$SL_2(\mathbb{Z})$  is generated by the two matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

To verify that  $\text{Ell}(M, W, \tau, z)$  satisfies the required transformation laws, it suffices to show the following four identities

$$(3.1) \quad \text{Ell}(M, W, \tau + 1, z) = \text{Ell}(M, W, \tau, z),$$

$$(3.2) \quad \text{Ell}(M, W, \tau, z + 1) = (-1)^l \text{Ell}(M, W, \tau, z),$$

$$(3.3) \quad \text{Ell}(M, W, \tau, z + \tau) = (-1)^l \exp(-\pi\sqrt{-1}l(\tau + 2z)) \text{Ell}(M, W, \tau, z),$$

$$(3.4) \quad \text{Ell}(M, W, -\frac{1}{\tau}, \frac{z}{\tau}) = \tau^{d-l} \exp\left(\frac{\pi\sqrt{-1}lz^2}{\tau}\right) \text{Ell}(M, W, \tau, z).$$

For Dedekind eta function  $\eta(\tau)$  and Jacobi-theta function  $\theta(\tau, z)$  we have the following transformation laws ([5]):

$$\begin{aligned}\eta^3(-\frac{1}{\tau}) &= (\frac{\tau}{\sqrt{-1}})^{\frac{3}{2}} \eta^3(\tau), & \eta^3(\tau+1) &= e^{\frac{\pi\sqrt{-1}}{4}} \eta^3(\tau), \\ \theta(\tau, z+1) &= -\theta(\tau, z), & \theta(\tau, z+\tau) &= -q^{-\frac{1}{2}} \exp(-2\pi\sqrt{-1}z) \theta(\tau, z), \\ \theta(\tau+1, z) &= \exp(\frac{\pi\sqrt{-1}}{4}) \theta(\tau, z), & \theta(-\frac{1}{\tau}, z) &= -\sqrt{-1} (\frac{\tau}{\sqrt{-1}})^{\frac{1}{2}} \exp(\pi\sqrt{-1}\tau z^2) \theta(\tau, \tau z).\end{aligned}$$

The first three identities (3.1), (3.2) and (3.3) are easy to verify by using the above-listed transformation laws. Here we only need to check (3.4) carefully. Indeed,

$$\begin{aligned}(3.5) \quad & \prod_{i=1}^d \theta(-\frac{1}{\tau}, x_i) \\ &= \prod_{i=1}^d -\sqrt{-1} (\frac{\tau}{\sqrt{-1}})^{\frac{1}{2}} \exp(\pi\sqrt{-1}\tau x_i^2) \theta(\tau, \tau x_i) \\ &= \exp(\frac{\tau p_1(M)}{4\pi\sqrt{-1}}) \prod_{i=1}^d -\sqrt{-1} (\frac{\tau}{\sqrt{-1}})^{\frac{1}{2}} \theta(\tau, \tau x_i).\end{aligned}$$

Here we use the assumption that

$$p_1(M) = \sum_{i=1}^d (2\pi\sqrt{-1}x_i)^2.$$

Similarly,

$$\begin{aligned}(3.6) \quad & \prod_{j=1}^l \theta(-\frac{1}{\tau}, w_j - \frac{z}{\tau}) \\ &= \prod_{j=1}^l -\sqrt{-1} (\frac{\tau}{\sqrt{-1}})^{\frac{1}{2}} \exp(\pi\sqrt{-1}\tau(w_j - \frac{z}{\tau})^2) \theta(\tau, \tau w_j - z) \\ &= \exp(\frac{\tau p_1(W)}{4\pi\sqrt{-1}} + \frac{\pi\sqrt{-1}lz^2}{\tau}) \prod_{j=1}^l -\sqrt{-1} (\frac{\tau}{\sqrt{-1}})^{\frac{1}{2}} \theta(\tau, \tau w_j - z).\end{aligned}$$

In the last equality we use the assumption that

$$c_1(W) = \sum_{j=1}^l 2\pi\sqrt{-1}w_j = 0.$$

Combining the transformation law of  $\eta(\tau)$ , (3.5), (3.6) and the fact that  $p_1(M) = p_1(W)$  leads to

$$\begin{aligned}
& \text{Ell}(M, W, -\frac{1}{\tau}, \frac{z}{\tau}) \\
&= \int_M \left[ \exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \left(\eta\left(-\frac{1}{\tau}\right)\right)^{3(d-l)} \prod_{i=1}^d \frac{2\pi\sqrt{-1}x_i}{\theta\left(-\frac{1}{\tau}, x_i\right)} \prod_{j=1}^l \theta\left(-\frac{1}{\tau}, w_j - \frac{z}{\tau}\right) \right] \\
&= \tau^{d-l} \exp\left(\frac{\pi\sqrt{-1}lz^2}{\tau}\right) \int_M \left[ \exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \left(\eta(\tau)\right)^{3(d-l)} \prod_{i=1}^d \frac{2\pi\sqrt{-1}x_i}{\theta(\tau, \tau x_i)} \prod_{j=1}^l \theta(\tau, \tau w_j - z) \right] \\
&= \tau^{-l} \exp\left(\frac{\pi\sqrt{-1}lz^2}{\tau}\right) \int_M \left[ \exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \left(\eta(\tau)\right)^{3(d-l)} \prod_{i=1}^d \frac{2\pi\sqrt{-1}(\tau x_i)}{\theta(\tau, \tau x_i)} \prod_{j=1}^l \theta(\tau, \tau w_j - z) \right] \\
&= \tau^{d-l} \exp\left(\frac{\pi\sqrt{-1}lz^2}{\tau}\right) \int_M \left[ \exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \left(\eta(\tau)\right)^{3(d-l)} \prod_{i=1}^d \frac{2\pi\sqrt{-1}x_i}{\theta(\tau, x_i)} \prod_{j=1}^l \theta(\tau, w_j - z) \right] \\
&= \tau^{d-l} \exp\left(\frac{\pi\sqrt{-1}lz^2}{\tau}\right) \text{Ell}(M, W, \tau, z)
\end{aligned}$$

The last but one equality is due to the fact that in the integrand we are only concerned with the homogeneous part of degree  $d$  ( $\deg(x_i) = \deg(w_j) = 1$ ). This completes the proof of Theorem 3.2.

**3.2. Algebraic preliminaries.** Before discussing the arithmetic properties of the generalized elliptic genus  $\text{Ell}(M, W, \tau, z)$ , we need to review a well-known manipulation in algebraic number theory of how to derive modular forms from Jacobi forms.

Recall that the *Eisenstein series*  $G_{2k}(\tau)$  are defined to be ([17, p. 131])

$$G_{2k}(\tau) := -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) \cdot q^n,$$

where

$$\sigma_k(n) := \sum_{m > 0, m|n} m^k$$

and  $B_{2k}$  are the Bernoulli numbers.

These  $G_{2k}(\tau)$  carry rich arithmetic information. It is well-known that  $G_{2k}(\tau)$  ( $k \geq 2$ ) are modular forms of weight  $2k$  over the full modular group  $SL_2(\mathbb{Z})$  and the whole graded ring of modular forms over  $SL_2(\mathbb{Z})$  are generated by  $G_4(\tau)$  and  $G_6(\tau)$ . However,  $G_2(\tau)$  is *not* a modular form but called *quasi-modular form* as it transforms as follows ([17, p. 138]).

$$(3.7) \quad G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - \frac{c(c\tau + d)}{4\pi\sqrt{-1}}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

The following proposition, which is a well-known fact in algebraic number theory and has been used implicitly by Gritsenko in the proof of [10, Lemma 1.6], provides us with a method for deriving modular forms from Jacobi forms.

**Proposition 3.5.** Suppose a function  $\varphi(\tau, z) : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$  satisfies

$$(3.8) \quad \varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \exp\left(\frac{2\pi\sqrt{-1}mcz^2}{c\tau + d}\right) \cdot \varphi(\tau, z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

i.e.,  $\varphi(\tau, z)$  transforms like a Jacobi form of weight  $k$  and index  $m$ .

Then, if we define

$$\Phi(\tau, z) := \exp(-8\pi^2 m G_2(\tau) z^2) \varphi(\tau, z),$$

we have

$$(3.9) \quad \Phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \Phi(\tau, z).$$

This means, if we set

$$\Phi(\tau, z) =: \sum_{n \in \mathbb{Z}} a_n(\tau) \cdot z^n,$$

then

$$a_n\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{k+n} a_n(\tau).$$

In particular, if  $\varphi(\tau, z)$  is a weak Jacobi form of weight  $k$  and index  $m$ , then these  $a_n(\tau)$  are modular forms of weight  $k + n$  over  $SL_2(\mathbb{Z})$ .

*Proof.* (3.9) can be verified directly by using the assumption condition (3.8) and the transformation law (3.7). If moreover  $\varphi(\tau, z)$  is a weak Jacobi form, then  $\varphi(\tau, z)$  and thus  $\Phi(\tau, z)$  are holomorphic and have no negative powers of  $q$  when considering their Fourier expansions in terms of  $q$  and  $y$ . This implies that these  $a_n(\tau)$  are also holomorphic and have no negative powers of  $q$  when considering the Fourier expansions of  $q$ , which gives the desired proof.  $\square$

Now with the assumptions in the Theorem 3.2 understood, we know that  $\text{Ell}(M, W, \tau, z)$  is a weak Jacobi form of weight  $d - l$  and index  $\frac{l}{2}$ . Then Proposition 3.5 tells us that

**Proposition 3.6.** The series  $a_n(M, W, \tau)$  determined by

$$\exp[l \cdot G_2(\tau) \cdot (2\pi\sqrt{-1}z)^2] \cdot \text{Ell}(M, W, \tau, z) =: \sum_{n \geq 0} a_n(M, W, \tau) \cdot (2\pi\sqrt{-1}z)^n$$

are modular forms of weight  $d - l + n$  over  $SL_2(\mathbb{Z})$ . Furthermore, the first three series of  $a_n(M, W, \tau)$  are of the following form:

$$\begin{aligned} & a_0(M, W, \tau) \\ &= \chi(M, \Lambda_{-1}W^*) + q \cdot \chi\left(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)\right) + q^2 \cdot (\cdots), \\ & a_1(M, W, \tau) = \sum_{p=0}^l (-1)^p \left(p - \frac{l}{2}\right) \chi(M, \Lambda^p W^*) + q \cdot (\cdots), \\ & a_2(M, W, \tau) = \left[-\frac{l}{24} \chi(M, \Lambda_{-1}W^*) + \frac{1}{2} \sum_{p=0}^l (-1)^p \left(p - \frac{l}{2}\right)^2 \chi(M, \Lambda^p W^*)\right] + q \cdot (\cdots). \end{aligned}$$

*Proof.* The first statement is a direct application of Proposition 3.5 as  $\text{Ell}(M, W, \tau, z)$  is a weak Jacobi form of weight  $d - l$  and index  $\frac{l}{2}$ . For the second one, if we set

$$\exp[lG_2(\tau)(2\pi\sqrt{-1}z)^2] =: A_0(y) + A_1(y) \cdot q + (\cdots) \cdot q^2,$$

and

$$\text{Ell}(M, W, \tau, z) =: B_0(y) + B_1(y) \cdot q + (\cdots) \cdot q^2,$$

we can easily deduce from their explicit expressions that

$$\begin{aligned} A_0(y) &= \exp\left[-\frac{l}{24}(2\pi\sqrt{-1}z)^2\right] = 1 - \frac{l}{24}(2\pi\sqrt{-1}z)^2 + \cdots, \\ A_1(y) &= l(2\pi\sqrt{-1}z)^2 - \frac{l^2}{24}(2\pi\sqrt{-1}z)^4 + \cdots, \\ B_0(y) &= \sum_{p=0}^l (-1)^p \chi(M, \Lambda^p W^*) y^{p-\frac{l}{2}} \\ &= \sum_{p=0}^l (-1)^p \chi(M, \Lambda^p W^*) \left[1 + (p - \frac{l}{2})(2\pi\sqrt{-1}z) + \frac{1}{2}(p - \frac{l}{2})^2(2\pi\sqrt{-1}z)^2 + \cdots\right], \\ B_1(y) &= \chi\left(M, \Lambda_{-1} W^* \otimes (-2(d-l) - W - W^* + T + T^*)\right) + 2\pi\sqrt{-1}z(\cdots). \end{aligned}$$

Note that

$$\sum_{n \geq 0} a_n(M, W, \tau) (2\pi\sqrt{-1}z)^n = A_0(y)B_0(y) + [A_0(y)B_1(y) + A_1(y)B_0(y)]q + \cdots$$

then it is easy to deduce the expressions in our Proposition 3.6 in terms of those of  $A_0(y)$ ,  $A_1(y)$ ,  $B_0(y)$  and  $B_1(y)$ .  $\square$

**3.3. -1-phenomenon of the generalized elliptic genus.** In this subsection, via Proposition 3.6 presented in the last subsection, we will investigate the arithmetic information of the generalized elliptic genus  $\text{Ell}(M, W, \tau, z)$ , which can be viewed as an appropriate -1-phenomenon of  $\text{Ell}(M, W, \tau, z)$ .

We will present one proposition and two examples related to  $a_2(M, W, \tau)$ ,  $a_0(M, W, \tau)$  and  $a_1(M, W, \tau)$  respectively to illustrate an appropriate -1-phenomenon of the generalized elliptic genus  $\text{Ell}(M, W, \tau, z)$ .

Our next proposition related to  $a_2(M, W, \tau)$  gives the “reason” why these  $a_n(M, W, \tau)$  should be the -1-phenomenon of  $\text{Ell}(M, W, \tau, z)$ .

**Proposition 3.7.**  $a_2(M, W, \tau)$  is a modular form of weight  $d - l + 2$  over  $SL_2(\mathbb{Z})$  provided that  $p_1(M) = p_1(W)$  and  $c_1(W) = 0$  in  $H^*(M, \mathbb{R})$ . Consequently, if either (i)  $d - l$  is odd, or (ii)  $d \leq l$  but  $d - l \neq -2$ , we have

$$(3.10) \quad \sum_{p=0}^l (-1)^p (p - \frac{l}{2})^2 \chi(M, \Lambda^p W^*) = \frac{l}{12} \chi(M, \Lambda_{-1} W^*).$$

Moreover, if  $W = T$  and  $c_1(M) = 0$  in  $H^*(M, \mathbb{R})$ , (3.10) is nothing but the original -1-phenomenon of the Hirzebruch  $\chi_y$ -genus.

*Proof.* If either (i)  $d - l$  is odd or (ii)  $d \leq l$  but  $d - l \neq -2$ ,  $a_2(M, W, \tau)$  is a modular form over  $SL_2(\mathbb{Z})$  whose weight is either (i) odd or (ii) no more than 2 but not zero. This means  $a_2(M, W, \tau) \equiv 0$  and then its expression in Proposition 3.6 gives (3.10).

If  $W = T$ , then

$$\text{the right-hand side of (3.10)} = \frac{d}{12} \chi(M, \Lambda_{-1} T^*) = \frac{d}{12} \chi_y(M) \Big|_{y=-1} = \frac{d}{12} c_d(M).$$

However,

$$\begin{aligned} & \text{the left-hand side of (3.10)} \\ &= \sum_{p=0}^d (-1)^p \left(p - \frac{d}{2}\right)^2 \chi^p(M) \\ &= \sum_{p=0}^d (-1)^p \left[2 \cdot \frac{p(p-1)}{2} + (1-d)p + \frac{d^2}{4}\right] \chi^p(M) \\ &= 2a_2(M) - (1-d)a_1(M) + \frac{d^2}{4}a_0(M) \\ &= \frac{d(3d-5)}{12}c_d(M) + \frac{(1-d)d}{2}c_d(M) + \frac{d^2}{4}c_d(M) \quad (\text{via (1.3) and } c_1(M) = 0) \\ &= \frac{d}{12}c_d(M) \\ &= \text{the right-hand side of (3.10)}. \end{aligned}$$

□

The last two examples related to  $a_0(M, W, \tau)$  and  $a_1(M, W, \tau)$  give much arithmetic information of  $M$  and  $W$ .

**Example 3.8.** By Proposition 3.6 we know that  $a_0(M, W, \tau)$  is a modular form of weight  $d - l$  over  $SL_2(\mathbb{Z})$  provided that  $p_1(M) = p_1(W)$  and  $c_1(M) = 0$  in  $H^2(M, \mathbb{R})$ . Consequently,

- (1) if either  $d - l$  is odd or  $d - l \leq 2$  but is nonzero, we have

$$\chi(M, \Lambda_{-1} W^*) = \chi\left(M, \Lambda_{-1} W^* \otimes (-2(d-l) - W - W^* + T + T^*)\right) = 0;$$

- (2) if  $d - l = 4$ ,  $a_0(M, W, \tau)$  is proportional to the Eisenstein series

$$G_4(\tau) = -\frac{B_4}{8} + q + \cdots = \frac{1}{240} + q + \cdots$$

and so

$$\chi\left(M, \Lambda_{-1} W^* \otimes (-2(d-l) - W - W^* + T + T^*)\right) = 240\chi(M, \Lambda_{-1} W^*);$$

- (3) if  $d - l = 6$ ,  $a_0(M, W, \tau)$  is proportional to the Eisenstein series

$$G_6(\tau) = -\frac{B_6}{12} + q + \cdots = -\frac{1}{504} + q + \cdots$$

and so

$$\chi\left(M, \Lambda_{-1} W^* \otimes (-2(d-l) - W - W^* + T + T^*)\right) = -504\chi(M, \Lambda_{-1} W^*);$$



(4) if  $d - l = 8$ ,  $a_0(M, W, \tau)$  is proportional to

$$[G_4(\tau)]^2 = \left[\frac{1}{240} + q + \cdots\right]^2 = \frac{1}{240^2} + \frac{1}{120}q + \cdots$$

and so

$$\chi\left(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)\right) = 480\chi(M, \Lambda_{-1}W^*).$$

**Example 3.9.** By Proposition 3.6 we know that  $a_1(M, W, \tau)$  is a modular form of weight  $d-l+1$  over  $SL_2(\mathbb{Z})$  provided that  $p_1(M) = p_1(W)$  and  $c_1(M) = 0$  in  $H^2(M, \mathbb{R})$ . Consequently, if either  $d - l$  is even or  $d - l \leq 1$  but  $d - l \neq -1$ , we have

$$\sum_{p=0}^l (-1)^p \left(p - \frac{l}{2}\right) \chi(M, \Lambda^p W^*) = 0.$$

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